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FIND RELATIONS IN FUZZY PAIRWISE-*T*2 BITOPOLOGICAL SPACE AND OTHER SUCH SPACES

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ABSTRACT

In mathematics, a bitopological space is a set endowed with two topologies. Typically, if the set is and the topologies are and then the bitopological space is referred to as . In this paper, we introduce a notion of fuzzy pairwise-$T2$ bitopological space and find relations with other such spaces. We also study some other properties of these concepts.

Keywords: Fuzzy Bitopological spaces; Q-neighbourhood; Fuzzy pairwise-$T2$ bitopological spaces.

INTRODUCTION

Bitopological variants of topological properties

Corresponding to well-known properties of topological spaces, there are versions for bitopological spaces.

- A bitopological space (X,τ_1,τ_2) is **pairwise compact** if each cover $\{U_i|i\in I\}$ of X with $U_i\in\tau_1\cup\tau_2$, contains a finite subcover. In this case, $\{U_i|i\in I\}$ must contain at least one member from τ_1 and at least one member from τ_2
- A bitopological space (X,τ_1,τ_2) is **pairwise Hausdorff** if for any two distinct points $x,y\in X$ there exist disjoint $U_1\in\tau_1$ and $U_2\in\tau_2$ with $x\in U_1$ and $y\in U_2$.
- A bitopological space (X,τ_1,τ_2) is **pairwise zero-dimensional** if opens in (X,τ_1) which are closed in (X,τ_2) form a basis for (X,τ_1), and opens in (X,τ_2) which are closed in (X,τ_1) form a basis for (X,τ_2).
- A bitopological space (X,σ,τ) is called **binormal** if for every F_σ σ-closed and F_τ τ-closed sets there are G_σ σ-open and G_τ τ-open sets such that $F_\sigma\subseteq G_\tau$ $F_\tau\subseteq G_\sigma$, and $G_\sigma\cap G_\tau=\emptyset$.

The notion of bitopological spaces was initially introduced by Kelly [7] in 1963. Concept of fuzzy pairwise-$T2$ (in short $FPT2$) bitopological spaces were introduced earlier by Kandil and El-Shafee [5]. Later on several other authors continued investigating such concepts. Fuzzy pairwise-$T2$ separation axioms have also been introduced by Abu Sufiya et al. [1] and Nouh [9]. The purpose of this paper is to introduce a definition of fuzzy pairwise-$T2$ bitopological space and derive some related results in this area. Also, we investigate that this concept holds good extension property in the sense of due to Lowen [1-4].

PRELIMINARIES ON FUZZY PAIRWISE-*T*2 BITOPOLOGICAL SPACES

Now we recall some definitions and concepts which will be used in our work.

Definition2.1. A fuzzy set μ in a set X is a function from X into the closed unit interval I=[0,1]. For every $x\in X,(x)\in I$ is called the grade of membership of x. Throughout this paper, IX will denote the set of all fuzzy sets from X into the closed unit interval I.

Definition 2.2. Let f be a mapping from a set X into a set Y and u be a fuzzy set in X. Then the image of u, written as $f(u)$, is a fuzzy set in Y whose membership function is given by $f(u)(y)$={ sup{u (X)} if f $^{-1}$ [{x}] □ □ and 0 for otherwise

Definition 2.3. Let f^{-1} be a mapping from a set X into a set Y and v be a fuzzy set in Y. Then the inverse of v written as $f^{-1}(v)$ is a fuzzy set in X which is defined by $f^{-1}(v)(x)=v(f(x))$, for $x\in X$.

Definition 2.4. A fuzzy set μ in X is called a fuzzy singleton if f $(x)=r,(0<r\leq1)$ for a certain $x\in X$ and $\mu(y)=0$ for all points y of X except x. The fuzzy singleton is denoted by xr and x is its support. We call xr is a fuzzy point if $0<r<1$. The class of all fuzzy singletons in X will be denoted by (X).

Definition 2.5. A fuzzy topology t on X is a collection of members of IX which is closed under arbitrary suprema and finite infima and which contains constant fuzzy sets 1 and 0. The pair (X,t) is called a fuzzy topological space (fts, in short) and members of t are called t-open (or simply open) fuzzy sets. A fuzzy set μ is called a t- closed (or simply closed) fuzzy set if $1-\mu\in t$.

Definition 2.6.] Let $(X,)$ and $(Y,)$ be two fuzzy topological spaces. A mapping f: $(X,)\longrightarrow(Y,s)$ is called an fuzzy continuous iff for every $v\in s$, $f-1(v)\in t$.

Definition 2.7. Let $(X,)$ and $(Y,)$ be two fuzzy topological spaces. A mapping f: $(X,)\longrightarrow(Y,s)$ is called an fuzzy open iff for every $u\in t$, $f(u)\in s$.

Definition 2.8. Let f be a real valued function on a topological space. If $\{:(x)>\alpha\}$ is open for every real $\in I1$, then f is called lower semi continuous function.

Definition 2.9. Let X be a nonempty set and T be a topology on X. Let $t=(T)$ be the set of all lower semi continuous functions from (X,T) to I (with usual topology). Thus $(T)=\{\mu\in IX$: $\mu-1(\alpha,1]\in T\}$ for each $\alpha\in I1$. It can be shown that (T) is a fuzzy topology on X.
Let P be a property of topological spaces and FP be its fuzzy topology analogue. Then FP is called a 'good extension' of P "iff the statement $(X,)$ has P iff $(X,(T))$ has FP" holds good for every topological space (X,T).

Definition 2.10. A fuzzy singleton xr is said to be quasi-coincident with a fuzzy set μ, denoted by $xrq\mu$ iff $r+(x)>1$. If xr is not quasi-coincident with μ, we write $xr\bar{q}\mu$.

Definition 2.11. A fuzzy set u of $(X,)$ is called quasi-neighborhood (Q-nbd, in short) of xr iff there exists $v\in t$ such that $xrqv$ and $v\subset u$. If xr is a fuzzy point or a fuzzy single tone, then $(xr,)=\{\mu\in t$: $xr\in\mu\}$ is the family of all fuzzy t-open neighborhoods (t-nbds, in short) of xr and $NQ(xr,t)=\{\mu\in t$: $xrq\ \mu\}$ is the family of all Q-neighborhoods (Q-nbd, in short) of xr.

Definition 2.12. A fuzzy bitopological space (fbts, in short) is a triple $(X,,)$ where s and t are arbitrary fuzzy topologies on X.
Definition 2.13. Let $(X,,)$ and $(Y,s1,t1)$ be two fuzzy bitopological spaces. A mapping f: $(X,s,t)\longrightarrow(Y,s,t)$ is called an fuzzy FP-continuous iff f: $(X,s)\longrightarrow(Y,s1)$ and f: $(X,t)\longrightarrow(Y,t1)$ are both continuous.

Definition 2.14. Let $(X,,)$ and $(Y,s1,t1)$ be two fuzzy bitopological spaces. A mapping f: $(X,s,t)\longrightarrow(Y,s,t)$ is called an fuzzy FP-open iff f: $(X,s)\longrightarrow(Y,s1)$ and f: $(X,t)\longrightarrow(Y,t1)$ are both open.

Definition 2.15. A space (X,S,T) is said to be pairwise Hausdorff iff for each two distinct points x and y, there are a S-neighbourhood U of x and a T-neighbourhood V of y such that $U\cap V=\emptyset$ [3-8].

FUZZY PAIRWISE T2-SPACES

Definition 3. 1. An fbts $(X,,)$ is called
(a) $FPT2(i)$ iff for every pair of fuzzy singletons xr,ys in X with $x\neq y$, there exist fuzzy sets $\mu\in s,\lambda\in t$ such that $xrq\mu$, $ysq\lambda$ and $\mu\cap\lambda=0$.

(b)[9] $FPT2(ii)$ iff $(\forall xr,ys\in S(X),x\neq y)$, $(\exists\mu\in N(xr,s)(\exists\ \lambda\in NQ(yr,t))$ $(\mu\bar{q}\lambda)$ or $(\exists\mu*\in N(xr,\ t)(\exists\ \lambda*\in NQ(yr,s))(\mu*\bar{q}\lambda*)$.

(c)[5] $FPT2(iii)$ iff for every pair of fuzzy singletons xp,yr in X such that $xp\bar{q}yr$, there exist fuzzy sets $\mu\in s,\lambda\in t$ such that $xp\in\mu$, $yr\in\lambda$ and $\mu\bar{q}\lambda$.

(d)[1] $FPT2(iv)$ iff for every pair of fuzzy singletons xr,ys in X with $x\neq y$, there exist fuzzy sets $\mu\in s,\lambda\in t$ such that $xr\in\mu$, $ys\in\lambda$ and $\mu\cap\lambda$=0.
(e)[1] $FPT2(v)$ iff for any two distinct fuzzy points xr,ys in X, there exist fuzzy sets $\mu\in s,\lambda\in t$ such that $xr\in\mu$, $ys\in\lambda$ and $\mu\subseteq\lambda c$.

Theorem 3.2. Let $(X,,)$ be an fbts. Then we have the following implications:
(a)⇔(d) ⇒ (b) ⇒ (e) but (b)⇏ (d), (e)⇏ (b), (a)⇏ (c) and (c)⇏ (a).

Proof: (a) ⇒ (d): Let $xr,\in S(X)$ with $x\neq y$. Since $(X,,)$ is FP$T2(i)$-space, then there exist fuzzy sets $\mu\in s,\lambda\in t$ such that $x1-rq\mu$, $y1-pq\lambda$ and $\mu\cap\lambda$=0.
That is, $(x)>r$, $(y)>p$ and $\mu\cap\lambda$=0.
So, $xr\in\mu$, $yp\in\lambda$ and $\mu\cap\lambda$=0. Hence $(X,,)$ is FP$T2(iv)$-space. Similarly we can show that (d) ⇒ (a).

(d) ⇒ (b): Let $xr,\in S(X)$ with $x\neq y$. Choose $p*\in(0,1)$ such that $p*>1-p$. Since $(X,,)$ is FP$T2(iv)$-space, then there exist fuzzy sets $\mu\in s,\lambda\in t$ such that $xr\in\mu$, $yp*\in\lambda$ and $\mu\cap\lambda$=0. That is, $xr\in\mu$, $(y)\geq p*$ and $\mu\bar{q}\lambda$.
Since $(y)\geq p*$ and $p*>1-p$, then we have
$(y)>1-p \Rightarrow (y)+p>1$. So, $ypq\lambda$.
Hence $xr\in\mu$, $ypq\lambda$ and $\mu\bar{q}\lambda$. Therefore $(X,,)$ is FP$T2(ii)$-space.

(b) ⇒ (e): Let xr, be two distinct fuzzy points in X. Since $(X,,)$ is FP$T2(ii)$-space, then there exist fuzzy sets $\mu\in s,\lambda\in t$ such that $xr\in\mu$, $y1-pq\lambda$ and $\mu\bar{q}\lambda$.
That is, $xr\in\mu$, $(y)+1-p>1$ and $\mu\subseteq\lambda c$.
That is, $xr\in\mu$, $(y)>p$ and $\mu\subseteq\lambda c$. That is, $xr\in\mu$, $yp\in\lambda$ and $\mu\subseteq\lambda c$. Hence $(X,,)$ is FP$T2(v)$.

Theorem 3.5. If an fbts $(X,,)$ is $FPT2(j)$, then $(X,s\cup t)$ is $FPT2(j)$, where j=I, ii, iii, iv, v.
Proof: Obvious.
The converse of the above theorem 3.5 is not true in general.

Theorem 3.7. Let $(X,,)$ be a fuzzy bitopological space, $A\subset X$ and $sA=\{u/A : u\in s\}$, $tA=\{v/A : v\in t\}$. Then
(a) $(X,,)$ is $FPT2(i) \Rightarrow (A,sA,tA)$ is $FPT2(i)$.
(b) $(X,,)$ is $FPT2(ii) \Rightarrow (A,sA,tA)$ is $FPT2(ii)$.
(c) $(X,,)$ is $FPT2(iii) \Rightarrow (A,sA,tA)$ is $FPT2(iii)$.
(d) $(X,,)$ is $FPT2(iv) \Rightarrow (A,sA,tA)$ is $FPT2(iv)$.
(e) $(X,,)$ is $FPT2(v) \Rightarrow (A,sA,tA)$ is $FPT2(v)$.

Proof: (a) Suppose $(X,,)$ is $FPT2(i)$. We have to show that $(A,,tA)$ is $FPT2(i)$. Let $xr,\in S(A)$ with x$\neq y$. Then $xr,\in S(X)$ with x$\neq y$. Since $(X,,)$ is $FPT2(i)$, then there exist fuzzy sets $\mu\in s,\lambda\in t$ such that $xrq\mu$, $ysq\lambda$ and $\mu\cap\lambda$=0. Now it is clear that $\mu/A\in sA$, $\lambda/A\in tA$ for every $\mu\in s,\in t$ respectively.
Now, $xrq\mu$, $ysq\lambda$ implies that $(x)+r>1$ and $\lambda(y)+s>1$.
But, $(\mu/A)(x)=\mu(x)$ and $(\lambda/A)(y)=\lambda(y)$. Then $(\mu/A)(x)+r>1$ and $(\lambda/A)(y)+s>1$. So, $x(\mu/A)$, $ysq(\lambda/A)$.
Also, $(\mu/A)\cap(\lambda/A)=(\mu\cap\lambda)/A=0$, since $\mu\cap\lambda$=0. Hence $(A,,tA)$ is $FPT2(i)$.
Proofs of (b), (c), (d) and (e) are similar.

Theorem 3.8. Let $(X,1,T2)$ be a bitopological space. Then
(a) $(X, T1,2)$ is $PT2 \Leftrightarrow (X,\omega(T1),\omega(T2))$ is $FPT2(i)$.
(b) $(X, T1,2)$ is $PT2 \Leftrightarrow (X,\omega(T1), \omega(T2))$ is $FPT2(ii)$.
(c) $(X,1,T2)$ is $PT2 \Leftrightarrow (X, \omega(T1), \omega(T2))$ is $FPT2(iii)$.
(d) $(X,1,T2)$ is $PT2 \Leftrightarrow (X, \omega(T1), \omega(T2))$ is $FPT2(iv)$.
(e) $(X,1,T2)$ is $PT2 \Leftrightarrow (X, \omega(T1), \omega(T2))$ is $FPT2(v)$.

Proof: (a) Suppose that $(X,1,T2)$ is $PT2$. We have to show that $(X,(T1),\omega(T2))$ is $FPT2(i)$. Let $xp,\in S(X)$ with $x\neq y$. Since $(X,1,T2)$ is $PT2$, then there exist $U\in T1$, $V\in T2$ such that $x\in U$, $y\in V$ and $U\cap V=\emptyset$. This implies $(1U\in(xp,\omega(T1)),(1V\in NQ(yr,\omega(T2))$ and $1V\cap 1U=0$. Hence $(X,(T1),\omega(T2))$ is $FPT2(i)$.

Conversely, suppose that $(X,(T1),\omega(T2))$ is $FPT2(i)$. We have to show that $(X,1,T2)$ is $PT2$. Let $x,\in X$ such that $x\neq y$. Since $(X,\omega(T1), \omega(T2))$ is $FPT2(i)$, then $(\exists\ \mu\in NQ(x1,\omega(T1))$, $(\exists\eta\in NQ(y1,\omega(T2))$ and $\mu\cap\ \eta=0$.
Now, $\mu\in NQ(x1,\omega(T1)),\eta\in NQ(y1,\omega(T2)$ implies that $\mu(x)+1>1$ and $\eta(y)+1>1$. That is, $(x)>0$ and $(y)>0$. Hence $x\in\mu-1(0,1]\in T1$, $y\in\eta-1(0,1]\in T2$.
To show that $\mu-1(0,1]\cap\eta-1(0,1]=0$, suppose that $\mu-1(0,1]\cap\eta-1(0,1]\neq 0$. Then there exists $z\in\mu-1(0,1]\cap\eta-1(0,1]$ such that $\mu(z)>0$ and $\eta(z)>0$. Consequently $(\mu\cap\ \eta)(z)\neq 0$ which contradicts the fact that $\mu\cap\ \eta=0$.
Proofs of (c) and (d) are similar and for the proof of (b), cf. [9].

Theorem 3.9. Given $\{(Xi,,ti):i\in\Lambda\}$ be a family of fuzzy bijtopological spaces. Then the product fbts $(\Pi Xi$, Πsi, $\Pi ti)$ is $FPT2(j)$ if each coordinate space (Xi,si,ti) is $FPT2(j)$, where $j=i,ii,iii,iv,v$.
Proof: Suppose each coordinate space $(Xi,,ti)$ is $FPT2(i)$. We shall show that the product space is $FPT2(i)$. Let xr, $ys\in(\Pi Xi)$ with $x\neq y$. Again suppose that $x=\Pi xi$, $y=\Pi yi$. Then $xi\neq yi$ for some $i\in\Lambda$, since $x\neq y$. Now consider (xi) ,$(yi)s\ \in S(Xi)$. Since $(Xi,,ti)$ is $FPT2(i)$, then there exist $\mu i\in si,\lambda i\in ti$ such that $(xi)r\ q\mu i$, $(yi)s\ q\lambda i$ and $\mu i\cap\lambda i=0$. Now consider $\mu=\Pi\mu j$ and $\lambda=\Pi\lambda j$, where $\mu i=\lambda i=1$ for $i\neq j$ and $\mu j=\mu j$, $\lambda j=\lambda j$. Then $\mu\in\Pi si$, $\lambda\in\Pi ti$ and we can easily show that $xrq\mu$, $ysq\lambda$ and $\mu\cap\lambda=0$. Hence the product space is $FPT2(i)$.
Other proofs are similar.

Theorem 3.10. A bijective mapping from an fts $(X,)$ to an fts $(Y,)$ preserves the value of a fuzzy singleton (fuzzy point).

Proof: Let cr be a fuzzy singleton in X. So, there exist a point $a\in Y$ such that $(c)=a$. Now $f(cr)(a)=f(cr)(f(c))=\sup cr(c)=cr(c)=r$, since f is bijective. Hence ar has same value as cr.

Theorem 3.11. Let $(X,,)$ and (Y, $s1,t1$) be two fuzzy bitopological spaces and let $f:X\rightarrow Y$ be bijective and FP-open. Then
(a) $(X,,)$ is $FPT2(i)\Rightarrow$ (Y, $s1,t1$) is $FPT2(i)$.
(b) $(X,,)$ is $FPT2(ii)\Rightarrow$ (Y, $s1,t1$) is $FPT2(ii)$.
(c) $(X,,)$ is $FPT2(iii)\Rightarrow$ (Y, $s1,t1$) is $FPT2(iii)$.
(d) $(X,,)$ is $FPT2(iv)\Rightarrow$ (Y, $s1,t1$) is $FPT2(iv)$.
(e) $(X,,)$ is $FPT2(v)\Rightarrow$ (Y, $s1,t1$) is $FPT2(v)$.

Proof: (a) Suppose $(X,,)$ is $FPT2(i)$. We shall show that (Y, $s1,1$) is $FPT2(i)$. Let $ar,\in S(Y)$ with $a\neq b$. Since f is bijective, then there exist distinct fuzzy singletons cr, in X such that $f(c)=a,f(d)=b$ and $c\neq d$. Again since $(X,,)$ is $FPT2(i)$, then there exist fuzzy sets $\mu,\in s$, $\lambda\in t$ such that $crq\ \mu,dqq\lambda$ and $\mu\cap\lambda=0$.
Now, $crq\ \mu,q\lambda$ implies that $\mu(c)+r>1$ and $\lambda(d)+q>1$.
But $f(\mu)(a)=f(\mu)(f(c))=\sup\mu(c)=\mu(c)$, since f is bijective. So $(\mu)(a)+r>1$, since $\mu(c)+r>1$. Hence $ar(\mu)$. Similarly, $bq(\lambda)$.
Also, $(\mu\cap\lambda)(a)=\sup(\mu\cap\lambda)(c)\ :f(c)=a$
$(\mu\cap\lambda)(b)=\sup(\mu\cap\lambda)(d)\ :f(d)=b$.
Hence $(\mu\cap\lambda)=0\Rightarrow f(\mu)\cap f(\lambda)=0$.
Since f is FP-open, then$(\mu)\ \in s1$, $f(\eta)\in t1$. Now, it is clear that there exist $f(\mu)\ \in s1$, $f(\eta)\in t1$ such that $arqf(\mu)$, $bqqf(\lambda)$ and $f(\mu)\cap f(\lambda)=0$. Hence (Y, $s1,1$) is $FPT2(i)$.
Similarly, (b), (c), (d) and (e) can be proved.

Theorem 3.12. Let $(X,,)$ and (Y, $s1,t1$) be two fuzzy bitopological spaces and $f:X\rightarrow Y$ be FP-continuous and bijective. Then
(a) (Y, $s1,1$) is $FPT2(i)\Rightarrow(X,s,t)$ is $FPT2(i)$.
(b) (Y, $s1,1$) is $FPT2(ii)\Rightarrow(X,s,t)$ is $FPT2(ii)$.
(c) (Y, $s1,1$) is $FPT2(iii)\Rightarrow(X,s,t)$ is $FPT2(iii)$.
(d) (Y, $s1,1$) is $FPT2(iv)\Rightarrow(X,s,t)$ is $FPT2(iv)$.

Proof: We shall prove (a) only.
Suppose (Y, s1,1) is $FPT2(i)$. We claim that $(X,,)$ is $FPT2(i)$. For this, let cr,$\in S(X)$ with $c\neq d$. Then there exist distinct fuzzy singletons ar, in Y such that $f(c)=a, f(d)=b$

and $a\neq b$, since f is one-one. Again since (Y, s1,1) is $FPT2(i)$, then there exist fuzzy sets $\mu,\in s$, $\lambda\in t$ such that $arq\mu$, $bqq\lambda$ and $\lambda\cap\mu=0$.
This implies that $(a)+r>1$, $\lambda(b)+q>1$ and $\lambda\cap\mu=0$.
That is, $\mu(f(c))+r>1$, $\lambda(f(d))+q>1$ and $f-1(\lambda\cap\mu)=0$.
That is, $f-1(\mu)(c)+r>1$, $f-1(\lambda)(d)+q>1$ and $f-1(\lambda)\cap f-1(\mu)=0$.
That is, $crqf-1(\mu)$, $dqqf-1(\lambda)$ and $f-1(\lambda)\cap f-1(\mu)=0$.
Since f is FP-continuous, then $f-1(\mu)\in s$, $f-1(\eta)\in t$. Now, we see that there exist $f-1(\mu)\in s$, $f-1(\eta)\in t$ such that $crqf-1(\mu)$, $dqqf-1(\lambda)$ and $f-1(\lambda)\cap f-1(\mu)=0$. Hence $(X,,)$ is $FPT2(i)$.

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